Complex Number Test

1. Solve $z^2 - (3 - i)z + (2 - i) = 0$ and write your answer in $a + bi$ form, where $a, b \in \mathbb{R}$.

2. Find the modulus and argument of $z = \dfrac{\left(1+\sqrt{3}i\right)^5 (1-i)^4}{(1+i)^6}$, where $|z| > 0$, $-\pi \le \text{Arg } z < \pi$.

3. Given that $|z| = 1$, prove that $\left|\dfrac{z-\omega}{1-\overline{\omega}z}\right| = 1$, for any $\omega \in \mathbb{C}$.

(Hint : $|z|^2 = z\overline{z}$, for any $z \in \mathbb{C}$.)

4. Solve for θ : $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = i$.

5. Given that $|z - (1+2i)| = \text{Re}(z - 2 + 3i)$. Show that the locus of z in the Argand plane is a parabola and find the vertex of the parabola in complex number form .

6. **(a)** Let $z = \cos\theta + i\sin\theta$. Show that for all $n \in \mathbb{N}$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

(b) Show that $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$.

(c) Hence evaluate $\int (\cos^8\theta + \sin^8\theta)\,d\theta$.

7. Let $\dbinom{n}{k}$ be the coefficient of x^k in the binomial expansion of $(1 + x)^n$.

By expanding $(1 + i)^{2n}$, or otherwise, prove that

(i) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} = 2^n \cos\frac{n\pi}{2}$

(ii) $\displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} = 2^n \sin\frac{n\pi}{2}$.

Complex Number Test

1. $z = \dfrac{-[-(3-i)] \pm \sqrt{[-(3-i)]^2 - 4(1)(2-i)}}{2} = \dfrac{(3-i) \pm \sqrt{-2i}}{2}$

$\sqrt{-2i} = \sqrt{1-2(1)i+i^2} = \sqrt{[\pm(1-i)]^2} = \pm(1-i)$

Or Let $\sqrt{-2i} = x + yi \Rightarrow -2i = x^2 + 2xyi + (yi)^2 \Rightarrow -2i = (x^2 - y^2) + 2xyi$

$\begin{cases} x^2 - y^2 = 0 & \ldots(1) \\ xy = -1 & \ldots(2) \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \mp 1 \end{pmatrix} \Rightarrow x + yi = \pm(1-i)$

$\therefore \quad z = \dfrac{(3-i) \pm (1-i)}{2} = 2 - i \quad \text{or} \quad 1$

Or Use factorization: $z^2 - (3-i)z + (2-i) = [z-1][z-(2-i)] = 0 \Rightarrow z = 1$ or $2 - i$.

2. $\left|\dfrac{(1+\sqrt{3}i)^5(1-i)^4}{(1+i)^6}\right| = \dfrac{|1+\sqrt{3}i|^5|1-i|^4}{|1+i|^6} = \dfrac{2^5\sqrt{2}^4}{\sqrt{2}^6} = \underline{\underline{16}}$

$\operatorname{Arg}\left[\dfrac{(1+\sqrt{3}i)^5(1-i)^4}{(1+i)^6}\right] = 5\operatorname{Arg}(1+\sqrt{3}i) + 4\operatorname{Arg}(1-i) - 6\operatorname{Arg}(1+i) = 5\left(\dfrac{\pi}{3}\right) + 4\left(-\dfrac{\pi}{4}\right) - 6\left(\dfrac{\pi}{4}\right) = \underline{\underline{-\dfrac{5\pi}{6}}}$

3. $\left|\dfrac{z-\omega}{1-\bar{\omega}z}\right|^2 = \dfrac{z-\omega}{1-\bar{\omega}z}\overline{\dfrac{z-\omega}{1-\bar{\omega}z}} = \dfrac{(z-\omega)(\bar{z}-\bar{\omega})}{(1-\bar{\omega}z)(1-\omega\bar{z})} = \dfrac{|z|^2 - z\bar{\omega} - \bar{z}\omega + |\omega|^2}{1 - z\bar{\omega} - \bar{z}\omega + |\omega|^2|z|^2} = \dfrac{1 - z\bar{\omega} - \bar{z}\omega + |\omega|^2}{1 - z\bar{\omega} - \bar{z}\omega + |\omega|^2}$, since $|z| = 1$

$= 1$

Since $\left|\dfrac{z-\omega}{1-\bar{\omega}z}\right| > 0, \quad \therefore \left|\dfrac{z-\omega}{1-\bar{\omega}z}\right| = 1$

4. $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = i$

$(\operatorname{cis}\theta)(\operatorname{cis}\theta)^2 \ldots (\operatorname{cis}\theta)^n = \operatorname{cis}(\pi/2)$

$(\operatorname{cis}\theta)^{1+2+\ldots+n} = \operatorname{cis}(2k\pi + \pi/2)$, where $k \in \mathbb{Z}$.

$(\operatorname{cis}\theta)^{n(n+1)/2} = \operatorname{cis}(4k+1)\pi/2$

$\operatorname{cis}\left(\dfrac{n(n+1)}{2}\theta\right) = \operatorname{cis}\dfrac{(4k+1)\pi}{2}$

$\therefore \theta = \dfrac{(4k+1)\pi}{n(n+1)}$, where $k \in \mathbb{Z}$.

5. Put $z = x + yi$ in $|z - (1+2i)| = \operatorname{Re}(z - 2 + 3i)$

$|(x-1) + (y-2)i| = \operatorname{Re}[(x-2) + (y+3)i]$

$$\sqrt{(x-1)^2+(y-2)^2} = x-2$$
$$x^2-2x+1+(y-2)^2 = x^2-4x+4$$
$$(y-2)^2 = -2x+3$$
$(y-2)^2 = -4\left(\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$, which is the Cartesian equation of a parabola with vertex at $\left(\frac{3}{2}, 2\right)$.

Hence the locus of z is a parabola with vertex at $\frac{3}{2}+2i$.

Or Parabola : $x = -\frac{y^2}{2}+2y-\frac{1}{2}$. Vertex = $\left(-\frac{\Delta}{4a}, -\frac{b}{2a}\right) = \left(\frac{3}{2}, 2\right)$. Vertex = $\frac{3}{2}+2i$.

6. **(a)** $z = \cos\theta + i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta \qquad \dots\ (1)$$
$$\frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta} = \cos(-n\theta)+i\sin(-n\theta) = \cos n\theta - i\sin n\theta \qquad \dots\ (2)$$
$\frac{(1)+(2)}{2}$, $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $\frac{(1)-(2)}{2}$, $z^n - \frac{1}{z^n} = 2i\sin n\theta$

(b) $64(\cos^8\theta+\sin^8\theta) = 64\times\left\{\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^8+\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^8\right\}$

$$= \frac{1}{4}\times\left\{\left[z^8+8z^6+28z^4+56z^2+70+\frac{56}{z^2}+\frac{28}{z^4}+\frac{8}{z^2}+\frac{1}{z^8}\right]+\left[z^8-8z^6+28z^4-56z^2+70-\frac{56}{z^2}+\frac{28}{z^4}-\frac{8}{z^2}+\frac{1}{z^8}\right]\right\}$$
$$= \frac{1}{2}\times\left[z^8+28z^4+70+\frac{28}{z^4}+\frac{1}{z^8}\right] = \frac{1}{2}\left(z^8+\frac{1}{z^8}\right)+28\times\frac{1}{2}\left(z^4+\frac{1}{z^4}\right)+35$$
$$= \cos 8\theta + 28\cos 4\theta + 35$$

(c) $\int(\cos^8\theta+\sin^8\theta)d\theta = \frac{1}{64}\int(\cos 8\theta+28\cos 4\theta+35)d\theta = \frac{1}{64}\left[\frac{\sin 8\theta}{8}+7\sin 4\theta+35\theta\right]+c$

7. By the Binomial Theorem,

$$(1+i)^{2n} = \binom{2n}{0}+\binom{2n}{1}i+\binom{2n}{2}i^2+\binom{2n}{3}i^3+\dots+\binom{2n}{2n-1}i^{2n-1}+\binom{2n}{2n}i^{2n}$$
$$= \sum_{k=0}^{n}\binom{2n}{2k}i^{2k}+\sum_{k=0}^{n-1}\binom{2n}{2k+1}i^{2k+1} = \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}+i\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}i^{2k+1} \qquad \dots\ (1)$$

Also, by de Moivre’s Theorem,

$$(1+i)^{2n} = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{2n} = 2^n\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right) \qquad \dots\ (2)$$

By equating real and imaginary parts of (1) and (2),

(i) $\sum_{k=0}^{n}(-1)^k \binom{2n}{2k} = 2^n \cos\frac{n\pi}{2}$ **(ii)** $\sum_{k=0}^{n-1}(-1)^k \binom{2n}{2k+1} = 2^n \sin\frac{n\pi}{2}$.